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LINEAR RELATIONS AND CONGRUENCES FOR THE COEFFICIENTS OF DRINFELD MODULAR FORMS*

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ABSTRACT

We find congruences for the t-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$. We give generalized analogies of Siegel's classical observation on $SL_2(\mathbb{Z})$ by determining all the linear relations among the initial t-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$. As a consequence spaces M_k^0 are identified, in which there are congruences for the s-expansion coefficients.

1. introduction

Recently, Choie et al. [3] generalized a classical observation of Siegel [10] by determining all the linear relations among the initial Fourier coefficients of a modular forms on $SL_2(\mathbb{Z})$. As a consequence, they showed p-divisibility properties for Fourier coefficients of a modular form on $SL_2(\mathbb{Z})$. The author [1] investigated analogies of these results for a certain subspace of M_k^m which have a strong condition. Here, M_k^m is the vector space of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$ of weight k and type m. In this paper the author generalizes the result for the space M_k^m.

In Section 3, we find divisibility properties for t-expansion coefficients of Drinfeld modular forms in M_k^m (Theorem 3.1). As a consequence we obtain

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congruence relations of t-expansion coefficients of Drinfeld modular forms in M_k^m (Remark 3.5).

By using the action of the Hecke operators Gekeler [6] and López [9] proved the existence of congruences for the coefficients of two distinguished Drinfeld modular forms, the Poincaré series $P_{q+1,1}$ and the discriminant function Δ, respectively. Gallardo and López [4] showed that there exist congruences for the s-expansion coefficients of the Eisenstein series of weight $q^k - 1$, for any positive integer k.

In Section 4, combining the idea in [3] and [10] we find all the linear relations among the initial t-expansion coefficients of a Drinfeld modular form in M_k^0 (Theorem 4.1). As a consequence spaces M_k^0 are identified, in which there are congruences for the s-expansion coefficients.

Throughout, we adopt the following notation :

$\mathbb{F}_q$	finite field with q elements, of characteristic p
A	$\mathbb{F}_q[T]$, the ring of polynomials over $\mathbb{F}_q$
K	$\mathbb{F}_q(T)$, the rational function field over $\mathbb{F}_q$
K_∞	$\mathbb{F}_q((1/T))$, the completion of K at $1/T$
C	the completion of the algebraic closure of K_∞
Ω	$C - K_\infty$, the Drinfeld upper half plane

2. Preliminaries

Let $L = \tilde{\pi}A$ be the lattice in C corresponding to the Carlitz module ρ and e_L be the exponential function associated to L, i.e.,

$$e_L : C \to C, \quad e_L(z) := z \prod_{\lambda \in L - \{0\}} \big(1 - z/\lambda\big).$$

We define $t = t(z) := 1/e_L(\tilde{\pi}z)$ and $s = t^{q-1}$.

A Drinfeld modular form (for $GL_2(A)$) of weight k and type m (where $k \geq 0$ is an integer and m is a class in $\mathbb{Z}/(q-1)$) is a holomorphic function $f : \Omega \to C$ that satisfies:

(i) $f(\gamma z) = (\det\gamma)^{-m}(cz+d)^k f(z)$ for any $\gamma \in GL_2(A)$,

(ii) f is holomorphic at the cusp ∞.

Since Ω is connected as an analytic space, any Drinfeld modular form f is determined through its expansion around ∞. We, therefore, identify f with its

t-expansion

$$f = \sum_{i=0}^{\infty} a_f((q-1)i+m)t^{(q-1)i+m}.$$

Here and in what follows, we choose the representative m in the class with $0 \leq m < q-1$. Let M_k^m be the C-vector space of Drinfeld modular forms of weight k and type m. Throughout the article we suppose that $k \equiv 2m \mod(q-1)$ (if not, $M_k^m = \{0\}$).

For any integer $k \geq (q+1)m$, the vector spaces, $M^0_{k-(q+1)m}$ and M_k^m, are isomorphic via the isomorphism: $f \mapsto fh^m$, where h is the Poincaré series $P_{q+1,1}$ (see [7, page 681] for the definition of $P_{q+1,1}$). Indeed, we only have to show that the map defined in the above is surjective. Since the graded $C-$algebra $\bigoplus_{k',m'} M_{k'}^{m'}$ is the polynomial ring $C[g,h]$ and $0 \leq m < q-1$, any element w of M_k^m is $\sum_i c_i g^{a_i} h^{(q-1)b_i} h^m$ for some $c_i \in C$ and nonnegative integers a_i, b_i. Here, g is a Drinfeld modular form of weight $q-1$ and type 0. Then an element $\sum_i c_i g^{a_i} h^{(q-1)b_i}$ of $M^0_{k-(q+1)m}$ is the preimage of w by the map defined in the above.

Moreover, we know $\dim_C M_k^0 = [k/(q^2-1)] + 1$. Hence, we have

$$\dim_C M_k^m = \Big[\frac{k-(q+1)m}{q^2-1}\Big] + 1. \tag{2.1}$$

Indeed, If $k < (q+1)m$, then $M_k^m = \{0\}$ hence $\dim_C M_k^m = [\frac{k-(q+1)m}{q^2-1}] + 1$.

Gekeler [5, 4.2. Theorem] found a formula for the dimension of the vector space of finite modular forms of weight k and type m. Following the same method as [5, 4.2. Theorem] we can also obtain the formula (2.1), which agrees with Gekeler's. On the other hand, the formula (2.1) is reformulated as follows:

$$\dim_C M_k^m = \begin{cases} [\frac{k}{q^2-1}] + 1 & \text{if } k^* \geq m(q+1), \\ [\frac{k}{q^2-1}] & \text{otherwise,} \end{cases}$$

where $k^* \equiv k \mod(q^2-1)$ and $0 \leq k^* < q^2-1$.

For any $z \in \Omega$, we let $\Lambda_z = Az + A$, a rank 2 A-lattice in C. It induces a Drinfeld module ϕ^z of rank 2 determined by

$$\phi_T^z(X) = TX + g(z)X^q + \Delta(z)X^{q^2}.$$

The j-invariant $j(z)$ of ϕ^z is defined to be $g(z)^{q+1}/\Delta(z)$. The functions g and Δ in z are Drinfeld modular forms of weights $q-1$ and q^2-1, respectively. We normalize $g(z)$ and $\Delta(z)$ as follows

$$g_{\text{new}}(z) = \widetilde{\pi}^{1-q} g(z) \quad \text{and} \quad \Delta_{\text{new}}(z) = \widetilde{\pi}^{1-q^2} \Delta(z).$$

Hereafter we write $g(z)$ and $\Delta(z)$ for $g_{\text{new}}(z)$ and $\Delta_{\text{new}}(z)$, respectively. Then we have that $\Delta(z) = -h(z)^{q-1}$.

Let $r := \dim_C M_k^m$. For any $f \in M_k^m$, we define

$$W(f) = \frac{f}{g^\alpha h^{m+(q-1)(r-1)}},$$

where $\alpha := (k-2m)/(q-1) + (1-r)(q+1) - m \in \{0, 1, \ldots, q\}$.

PROPOSITION 2.1: *W is a vector space isomorphism from M_k^m onto the space R of polynomials in j of degree less than r.*

Proof. For $d = 0, 1, \ldots, r-1$, the products $j^d g^\alpha h^{m+(q-1)(r-1)}$ belong to M_k^m. Indeed, its t-expansion at ∞ shows that $j^d g^\alpha h^{m+(q-1)(r-1)}$ is analytic at ∞. Moreover, j, g and h are analytic on Ω. Thus the products $j^d g^\alpha h^{m+(q-1)(r-1)}$ belong to M_k^m.

Since $W(j^d g^\alpha h^{m+(q-1)(r-1)}) = j^d$, W carries the subspace Q of M_k^m generated by the Drinfeld modular forms $j^d g^\alpha h^{m+(q-1)(r-1)}$ isomorphically onto R. Hence, $\dim_C Q = r$ which implies $Q = M_k^m$. ∎

Let $\Gamma = GL_2(A)$. For any meromorphic Drinfeld modular form $G(z)$ of weight 2 and type 1, $\omega := G(z)\, dz$ is a 1-form on the compactification $\overline{\Gamma\backslash\Omega}$ of $\Gamma\backslash\Omega$. Let $G(z) = \sum_{n=n_0}^{\infty} a(n) t^n$ be the t-expansion of $G(z)$ at the cusp ∞. Let $\pi : \Omega \to \Gamma\backslash\Omega$ be the quotient map. Then we have

PROPOSITION 2.2: *(i) $Res_\infty \omega = a(1)/\tilde{\pi}$.*

(ii) $Res_\tau G(z)\, dz = Res_{\pi(\tau)} \omega$ for each $\tau \in \Omega$.

Proof. (i) follows from the simple fact that $-\tilde{\pi} t^2 dz = dt$. For any ordinary point $\tau \in \Omega$, (ii) is obvious. Suppose $\tau \in \Omega$ is an elliptic point. Let Γ_τ be the stabilizer of τ in Γ and $Z(K)$ be the center of scalar matrices. Let $e_\tau = |\Gamma_\tau/(\Gamma_\tau \cap Z(K))|$. Indeed, $e_\tau = q+1$ because τ is an elliptic point. We choose uniformizers x and y on Ω and $\Gamma\backslash\Omega$, respectively, with $x^{e_\tau} = y$. Then $dy = e_\tau x^{e_\tau - 1} dx = x^{e_\tau - 1} dx$, which gives the assertion (ii). ∎

3. Divisibility properties and congruences for coefficients of Drinfeld modular forms

In this section, we study congruences for t-expansion coefficients of Drinfeld modular forms belonging to M_k^m. The following theorem is motivated from the classical results [3] of p-divisibility properties for Fourier coefficients of modular forms on $SL_2(\mathbb{Z})$. These classical results play an important role in the p-adic theory of modular forms (see [8]). Unfortunately the author could not find an important role of Theorem 3.1 in the function field case. This requires a further research.

For a prime $\mathfrak{p}$ of degree 1, if $f_1 \equiv f_2 \not\equiv 0 \bmod \mathfrak{p}$ for $f_i \in M_{k_i}^{m_i} \cap A[[t]]$, then $k_1 \equiv k_2 \bmod (q-1)$ ([7, page 698]). This result leads us to study coefficients mod $\mathfrak{p}$ among Drinfeld modular forms whose weights are the same mod $(q-1)$. In addition, Theorem 3.1 gives mysterious congruence properties of Drinfeld modular forms. Throughout this section we let $r := \dim_C M_k^m$.

Theorem 3.1: *Let*

$$f = \sum_{i=0}^{\infty} a_f((q-1)i+m)t^{(q-1)i+m} \in M_k^m \cap A[[t]].$$

Then for any integer a satisfying that $p^a + 1 \geq m + r(q-1)$ and $m \equiv p^a + 1 \bmod (q-1)$, we have that

(i) *if $q > 2$, then*

$$a_f\left((q-1)\frac{p^a+1-m}{q-1} + m\right) \equiv 0 \bmod (T^q - T);$$

(ii) *if $q = 2$, then $a_f(2^a+1) \equiv -a_f(1) \bmod (T^q - T)$.*

Proof. We notice ([2, page 8]) that

$$\frac{dj}{dz} = -\widetilde{\pi}\frac{g^q}{h^{q-2}} \tag{3.1}$$

a meromorphic Drinfeld modular form of weight 2 and type 1. By Proposition 2.1 we have that for any nonnegative integer v, $j^v W(f)\frac{dj}{dz}$ is a meromorphic Drinfeld modular form of weight 2 and type 1 which is holomorphic on Ω. By Proposition 2.2 and the residue theorem ($\sum_{\mu \in \overline{\Gamma \backslash \Omega}} \mathrm{Res}_\mu (j^v W(f)\frac{dj}{dz})dz = 0$), the coefficient of t in

$$j^v W(f)\frac{dj}{dz}$$

vanishes.

Now we choose for v a particular non-negative integer l_a given by

$$l_a := \frac{p^a + 1 - m}{q-1} - r.$$

This makes the denominator of $j^{l_a} W(f)(dj/dz)$, a constant multiple of h^{p^a}.

The fact ([7, (6.11) Proposition]) that $g \equiv 1 \bmod (T^q - T)$, combined with (3.1) shows that the coefficient of t in

$$\frac{(-1)^{l_a+1}}{\widetilde{\pi}} j^{l_a} W(f) \frac{dj}{dz} = \frac{g^{(q+1)l_a - \alpha + q} f}{h^{(q-1)(r+l_a)+m-1}} \equiv \frac{f}{h^{p^a}} \bmod (T^q - T)$$
$$\equiv \left(\sum_{i=0}^{\infty} a_f((q-1)i + m) t^{(q-1)i+m} \right)$$
$$\times \left(\frac{(-1)^{p^a}}{t^{p^a}} + t^{((q-1)^2-1)p^a} + \cdots \right) \bmod (T^q - T)$$

is zero mod $(T^q - T)$, where $\cdots$ means "higher terms in t". This implies the assertion. ∎

Corollary 3.2: *Suppose that $k \equiv 2 \bmod p$. Then we have the following*

(i) *if $q > 2$, then*

$$a_f((q-1)\frac{p^a + 1 - m}{q-1} + m) \equiv 0 \bmod (T^q - T)^p,$$

(ii) *if $q = 2$, then $a_f(2^a + 1) \equiv -a_f(1) \bmod (T^q - T)^p$.*

Proof. We use the same notations in the proof of Theorem 3.1. By the assumption that $k \equiv 2 \bmod p$, we have that $(q+1)l_a - \alpha + q \equiv 0 \bmod p$. Hence we obtain that

$$\frac{(-1)^{l_a+1}}{\widetilde{\pi}} j^{l_a} W(f) \frac{dj}{dz} = \frac{g^{(q+1)l_a - \alpha + q} f}{h^{(q-1)(r+l_a)+m-1}} \equiv \frac{f}{h^{p^a}} \bmod (T^q - T)^p,$$

which implies the assertion. ∎

Example 3.3: In the case $q = 2$, for any $f = \sum_{i=0}^{\infty} a_f(i+m) t^i \in M_k^m \cap A[[t]]$ we have that for any integer a satisfying that $2^a + 1 \geq m + r$,

$$a_f(2^a + 1) \equiv -a_f(1) \bmod (T^2 - T).$$

Example 3.4: Let the Poincaré series $h := P_{q+1,1}$ have t-expansion as follows

$$h = \sum_{i=0}^{\infty} a_h((q-1)i + 1) t^{(q-1)i+1}.$$

Then the coefficient of $t^{(q-1)i+2}$ in h^2 is $\sum_{j=0}^{i} a_h((q-1)j+1)a_h((q-1)(i-j)+1)$. By Corollary 3.2 we have that for any multiple a of b ($q=p^b>2$),

$$\sum_{j=0}^{\frac{p^a+1-m}{q-1}} a_h((q-1)j+1)a_h\Big((q-1)\Big(\frac{p^a+1-m}{q-1}-j\Big)+1\Big) \equiv 0 \bmod (T^q-T)^q.$$

Remark 3.5: Theorem 3.1 is powerful in the case the type m is equal to 2. Using differential ∂_k (see [2, page 3]) and product we can change the type of Drinfeld modular forms. Then we can obtain congruence relations of t-expansion coefficients of Drinfeld modular forms by Theorem 3.1.

4. Linear relations between Drinfeld modular form coefficients

In this section, we give all the linear relations among the initial t-expansion coefficients of a Drinfeld modular form in M_k^0. Let

$$r := \dim_C M_k^0 \quad \text{and} \quad \alpha := \frac{k}{q-1} + (1-r)(q+1) \in \{0,1,\ldots,q\}.$$

For any integer $N \geq 0$, we let

$$L_{k,N} := \left\{ (c_0, c_1, \ldots, c_{r+N}) \in C^{r+N+1} : \begin{array}{c} \sum_{i=0}^{r+N} c_i a_f((q-1)i) = 0 \\ \forall f = \sum_{i=0}^{\infty} a_f((q-1)i)t^{(q-1)i} \in M_k^0 \end{array} \right\}$$

be the space of linear relations satisfied by the first $r+N+1$ t-expansion coefficients of all the forms $f \in M_k^0$. In his study of Hilbert modular forms, Siegel [10] determined the spaces $L_{k,0}$ defined analogously.

To state our result, for each Drinfeld modular form $u \in M^0_{(q^2-1)N}$, define numbers $b(k,N,u;i)$ by

$$\frac{g^{q-\alpha}}{h^{(q-1)(r+N)-1}}u = \sum_{i=0}^{r+N} b(k,N,u;i)t^{-(q-1)i+1} + \sum_{i=1}^{\infty} c(k,N,u;i)t^{(q-1)i+1}.$$

In this notation, we have the following theorem.

Theorem 4.1: *The map $\phi_{k,N} : M^0_{(q^2-1)N} \longrightarrow L_{k,N}$ defined by*

$$\phi_{k,N}(u) = (b(k,N,u;0), b(k,N,u;1), \ldots, b(k,N,u;r+N))$$

provides a linear isomorphism from $M^0_{(q^2-1)N}$ onto $L_{k,N}$.

Proof. Let

$$u \in M^0_{(q^2-1)N} \quad \text{and} \quad f = \sum_{i=0}^{\infty} a_f((q-1)i)t^{(q-1)i} \in M^0_k.$$

The coefficient $\sum_{i=0}^{r+N} b(k,N,u;i)a_f((q-1)i)$ of t in

$$\frac{fg^{q-\alpha}}{h^{(q-1)(r+N)-1}}u = -\frac{1}{\tilde{\pi}}W(fu)\frac{dj}{dz}$$

is zero. Therefore, the map $\phi_{k,N}$ is well-defined. Clearly, $\phi_{k,N}$ is linear. Suppose that $\phi_{k,N}(u) = 0$. This assumption implies that $W(fu)dj/dz$ has a zero at least of order 2 at ∞. Furthermore, $W(fu)dj/dz$ is holomorphic on Ω. Therefore, $ufg^{q-\alpha}/h^{(q-1)(r+N)-1}$ is a cusp form of weight 2 and type 1. Hence, $ufg^{q-\alpha}/h^{(q-1)(r+N)-1}$ is the zero function, which implies $u = 0$. Hence, $\phi_{k,N}$ is injective. Let $f_1, \ldots, f_r$ be a basis of M^0_k. Let A be a $r \times (N+1+r)$-matrix whose ith row consists of the initial $N+r+1$ coefficients of f_i. Then the null space of A is equal to $L_{k,N}$. A valance formula [7, (5.14)] shows that the rank of A is r. Consequently, the rank-nullity theorem implies that $\phi_{k,N}$ is surjective. ∎

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